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On the Brown-York quasilocal energy, gravitational charge, and black hole horizons

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Abstract

We study a recently proposed horizon defining identity for certain black hole spacetimes. It relates the difference of the Brown-York quasilocal energy and the Komar charge at the horizon to the total energy of the spacetime. The Brown-York quasilocal energy is evaluated for some specific choices of spacetime foliations. With a certain condition imposed on the matter distribution, we prove this identity for spherically symmetric static black hole solutions of general relativity. For these cases, we show that the identity can be derived from a Gauss-Codacci condition that any three-dimensional timelike boundary embedded around the hole must obey. We also demonstrate the validity of the identity in other cases by explicitly applying it to several static, non-static, asymptotically flat, and asymptotically non-flat black hole solutions. These include the asymptotically Friedmann-Robertson-Walker (FRW) solutions and the case of a black hole with a global monopole charge.

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The concept of a black hole horizon, ever since its birth with Lemaitre's demonstration of the non-singularity of the Schwarzschild horizon, has played a monumental role in the understanding of the causal structure of several spacetimes. It has been associated with many general relativistic theorems and laws of import, e.g., the singularity theorems [1], the black hole area theorem [2], and the laws of black hole mechanics. It plays an important role in Hawking's semiclassical calculation on the evaporation of a black hole [3] and is also related to its entropy [4]. Finally, although the horizon may not have any special significance in the frame of a freely falling observer, to an asymptotic inertial observer it behaves very much like a physical membrane (see, e.g., Ref. [5]).

The behavior of families of null geodesics in a given spacetime determines if a horizon exists in it. Moreover, it is known that the properties of such a family are affected by the matter stress-tensor through the Raychaudhuri equation [6]. This prompts one to ask if there exists a direct characterization of the black hole horizon in terms of the quasilocal energy or gravitational "charge" of bounded regions embedded in such spacetimes. To seek such a characterization will be the main aim of this paper.

It is well known that there are inherent difficulties in defining energy in general relativity (GR), essentially owing to its non-localizability. So far, considerable effort has been put in to formulate a satisfactory definition. For quasilocal energy, we shall here adopt the definition given by Brown and York [7] (henceforth referred to as BY), which can be summarized as follows: The system under consideration is a spatial three-surface Σ bounded by a two-surface B in a spacetime region that can be decomposed as a product of a spatial three-surface and a real line-interval representing time. The time evolution of the two-surface boundary B is the timelike three-surface boundary ${}^3\mathsf{B}$. When Σ is taken to intersect ${}^3\mathsf{B}$ orthogonally, the BY quasilocal energy is defined as:

$$E = \frac{1}{8\pi} \oint_{\mathsf{B}} d^2x \sqrt{\sigma}(K - K_0) \quad , \tag{1}$$

where σ is the determinant of the 2-metric on B, K is the trace of the extrinsic curvature of B, and K_0 is a reference term that is used to normalize the energy with respect to a reference spacetime, not necessarily flat. (Here, we employ geometrized units, with $G = 1 = c$.) Henceforth we will use the acronym "QLE" to denote the BY quasilocal energy.

To compute the QLE for asymptotically flat solutions, we will choose the reference spacetime to be Minkowski. In that case, K_0 is the trace of the extrinsic curvature of a two-dimensional surface embedded in flat spacetime, such that it is isometric to B. For asymptotically non-flat solutions, the Hawking-Horowitz prescription [8,9] will be employed to determine the reference spacetime.

For describing interactions other than gravity, one uses separate measures for the mass (or energy) of a particle and its charge (which determines its strength of coupling to the field). By contrast, in GR there is a single measure for both. Besides, the gravitational field of a particle itself contributes to its gravitational charge since it contains energy. It is this aspect of the gravitational field that has been the main cause of ambiguity in attempts at distinguishing charge from energy in GR. However, one possibility for defining gravitational charge would be to suitably adopt the Gauss law in Newtonian gravity to the case of GR [10]. For asymptotically flat, static spacetimes, this generalization is readily implemented: Let t^a be a timelike Killing vector field normalized in such a way that its norm approaches

unity at infinity. Then, for an isolated system, the total mass M within a two-surface B is the total force that must be exerted by an observer at infinity on a test matter with unit surface mass density spread on B, such that each point on B follows an orbit of a Killing vector such as t^a. Then,

$$M_c = \frac{1}{4\pi} \oint_{\mathsf{B}} \mathbf{g}.\mathbf{ds} \quad , \tag{2}$$

where $\mathbf{g} = -N\nabla(\ln N)$, $N = \sqrt{-t^a t_a}$, and the integral is taken over the closed 2-surface B. The norm of $\mathbf{g}$ evaluated at the horizon gives the surface gravity of the hole. This is how gravitational charge is intimately related to surface gravity (and, therefore, the temperature) of a black hole. Note that, for a suitable choice of foliation of the spacetime region of interest with spacelike hypersurfaces, one can choose a time coordinate such that N plays the role of lapse function. This is the choice that we will exercise below.

The above expression can be equivalently written as

$$M_c = -\frac{1}{8\pi} \oint_{\mathsf{B}} \epsilon_{abcd} \nabla^c t^b \quad , \tag{3}$$

where ϵ_{abcd} is the volume element on the spacetime. It is in fact the Komar mass [11]. Equation (3) defines a conserved (both in space and time) gravitational charge when the spacetime is vacuum and admits a timelike Killing vector. When either of these conditions is relaxed, M_c depends upon the location of B. In such a case it describes the *quasilocal* charge associated with the spatial volume bounded by B.

It is clear from the above definition of gravitational charge that it is the lapse function that determines it. On the other hand QLE is not at all sensitive to it and is instead determined entirely by the spatial metric. Hence, the measures of quasilocal charge and energy will, in general, be different. In what follows, by the energy and charge of a spacetime region, we shall mean the gravitational quasilocal energy and charge (references to the electric or magnetic charge of that region will be made explicitly).

Dadhich [12,13] has recently proposed a novel energetics characterization of the horizon of a black hole in asymptotically flat, spherically symmetric static (SSS) spacetimes. He proposed that its location is at that curvature-radius, r, at which the following identity holds:

$$E_{\mathcal{H}} - E_{\infty} = M_{\mathcal{H}} \quad , \tag{4}$$

where $E_{\mathcal{H}}$ is the QLE at the horizon, E_{∞} is the total energy of the spacetime, and $M_{\mathcal{H}}$ is the value of the gravitational charge, M_c, at the horizon. The physical interpretation of this identity is as follows. The gravitational charge essentially defines the strength of the leading Newtonian potential. In the Newtonian sense, therefore, it measures the strength of the gravitational pull exerted by a body. On the other hand, as shown by Eq. (1), the field energy, $E_{\infty} - E_{\mathcal{H}}$, is obtained completely from the spatial part of the metric. Its significance can be seen by noting that it is related to the spatial conformal factor (see, e.g., Ref. [14]) that arises in higher post-Newtonian orders in the expansion of the spacetime metric around a non-rotating matter distribution. It, therefore, measures the amount of "curvature of space" due to that distribution in the sense that the spatial components of the Riemann

curvature tensor (up to post-Newtonian order) are determined by it. Also, in the specific case of the Schwarzschild spacetime, Dadhich [15] has argued how the gravitational field energy is related to the curvature of space. The above identity implies that the horizon is a surface where the magnitude of the gravitational field energy equals the gravitational charge.

The BY quasilocal energy (1) does not depend on the choice of coordinates on the quasilocal two-surface B. It, however, depends on the choice of the foliation and B itself. Hence, as we show below, the above identity (4) holds only for specific foliations of such spacetimes.

We now prove that Eq. (4) can be derived from a local relation between the covariant derivative of the trace of the extrinsic curvature of the timelike three-boundary ${}^3\mathsf{B}$ and certain scalars associated with the Ricci tensor. We first show that this relation is essentially a consequence of a Gauss-Codacci embeddability condition on ${}^3\mathsf{B}$, which is automatically satisfied because of the assumption that the boundary ${}^3\mathsf{B}$ is embeddable in the spacetime. Below, we explicitly state the assumptions under which the proof holds. Later we will consider various other examples of black hole spacetimes, not necessarily static or asymptotically flat, for which the identity (4) is valid.

The first relation we require is the decomposition of the four-dimensional (4D) Ricci scalar into spatial and timelike components

$$\mathcal{R} = \gamma^{\mu\nu}\gamma^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta} + 2n^\mu n^\nu \mathcal{R}_{\mu\nu} \quad , \tag{5}$$

where $\gamma_{\mu\nu}$ is the 3-metric on ${}^3\mathsf{B}$ and n_μ is its spacelike normal with unit norm. (We shall follow the conventions of Ref. [16].) The Gauss-Codacci relation for the projection of the Riemann tensor onto ${}^3\mathsf{B}$ gives for the first term on the right-hand side (rhs) above:

$$\gamma^{\mu\nu}\gamma^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta} = R - \Theta^2 + \Theta_{\mu\nu}\Theta^{\mu\nu} \quad , \tag{6}$$

where R is the 3D Ricci scalar associated with ${}^3\mathsf{B}$ and Θ is its extrinsic curvature. On using the Ricci identity, $\mathcal{R}_{\mu\alpha\nu\beta}n^\beta = 2\nabla_{[\mu}\nabla_{\alpha]}n_\nu$, the second term on rhs of Eq. (5) gives

$$n^\mu n^\nu \mathcal{R}_{\mu\nu} = \Theta^2 - \Theta_{\mu\nu}\Theta^{\mu\nu} + \nabla_\mu(\Theta n^\mu + b^\mu) \quad , \tag{7}$$

where $b^\mu \equiv n^\nu \nabla_\nu n^\mu$. Using Eqs. (7) and (6) in the decomposition formula (5), we obtain

$$\nabla_\mu(\Theta n^\mu + b^\mu) + R = \mathcal{R} - n^\mu n^\nu \mathcal{R}_{\mu\nu} \, . \tag{8}$$

This is the essential relation we will require below.

We now formulate a condition on the matter distribution that is required for the identity, Eq. (4) to hold in asymptotically flat SSS spacetimes. Let Σ be a smooth hypersurface transverse to the timelike Killing vector field t^a, such that it passes through the bifurcation surface $\mathcal{H}$ of the horizon. Let $\mathcal{H}_\epsilon$ be a smooth, one parameter family of surfaces in Σ that approach $\mathcal{H}$ as $\epsilon \to 0$. We restrict our attention to the spacetime region exterior to the (outer) horizon and foliate it with a one-parameter family of spacelike hypersurfaces, Σ_t, such that for any value of t, the leaf Σ_t bears the properties of Σ. Let u^a be a unit timelike normal to Σ. Then, $t^a = Nu^a$, where N is the lapse function. We shall assume that the following condition on the Ricci tensor holds in the spacetimes of interest

$$u^\mu u^\nu \mathcal{R}_{\mu\nu} = \mathcal{R} - n^\mu n^\nu \mathcal{R}_{\mu\nu} \quad , \tag{9}$$

which is obeyed by the Kerr-Newman family of spacetimes. If we assume that the Einstein field equations hold, then this condition translates to

$$T_{\mu\nu} \left(u^\mu u^\nu + n^\mu n^\nu \right) = 0 \quad , \tag{10}$$

where n^μ is the unit normal to B such that they both lie in Σ. This condition constrains the type of matter allowed to exist in the spacetime outside the horizon. The need for such a condition can be understood by the fact that the identity (4) is not expected to hold for additions of arbitrary matter field distributions outside the horizon since it can alter E_∞. With this assumption, Eq. (8) becomes

$$\nabla_\mu (\Theta n^\mu + b^\mu) = u^\mu u^\nu \mathcal{R}_{\mu\nu} - R\,. \tag{11}$$

Thus, the vector field $[\Theta n^\mu + b^\mu]$ fails to be divergenceless in the presence of a non-vanishing rhs, which acts as its source.

We consider spacetimes that are asymptotically flat. This imposes a condition on the fall-off behavior of the metric components at spatial infinity i^0. Note that the metric of any SSS spacetime can be written as

$$ds^2 = -N^2 dt^2 + \lambda^{-2} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \quad , \tag{12}$$

where N and λ are dependent on r only. This line element is particularly suited to the choice of foliation above, in the sense that Σ_t is just a t =constant hypersurface. Also note that if the two-boundary B is taken to be a sphere of constant curvature-radius r, on any Σ_t, then evolving the points on it along the orbits of the Killing field t^a will generate a three-boundary ${}^3\mathsf{B}$ that is orthogonal to Σ_t wherever they meet. By asymptotic flatness, we shall mean that λ has the following fall-off behavior:

$$\lambda \to 1 + \lambda_1 r^{-1} + O(r^{-2}) \quad , \tag{13}$$

as $r \to \infty$. Here, λ_1 is an r-independent constants.

Let $\Sigma_{\mathcal{H}_\epsilon}$ denote the region on a Σ-type hypersurface that is bounded from the 'interior' by $\mathcal{H}_\epsilon$ and from the 'exterior' by a two-sphere at infinity. We multiply both sides of Eq. (11) by the lapse N and integrate over the volume of $\Sigma_{\mathcal{H}_\epsilon}$. After a simple rearrangement of terms, we get:

$$\int_{\Sigma_{\mathcal{H}_\epsilon}} d^3x \sqrt{-g}\, \nabla_\mu (\Theta n^\mu + b^\mu) + \int_{\Sigma_{\mathcal{H}_\epsilon}} d^3x \sqrt{-g}\, R = \int_{\Sigma_{\mathcal{H}_\epsilon}} d^3x\, \sqrt{h}\, \mathcal{R}_{\mu\nu} u^\mu t^\nu \quad , \tag{14}$$

where h is the determinant of the three-metric on $\Sigma_{\mathcal{H}_\epsilon}$. The term on the rhs is the simplest to interpret. When divided by 4π, it is just the Komar mass and, therefore, contributes $(E_\infty - M_{\mathcal{H}_\epsilon})$, where $M_{\mathcal{H}_\epsilon}$ is the gravitational charge at $\mathcal{H}_\epsilon$. Here, we have implicitly used the fact that $E_\infty = (M_c)_\infty$. This is justified since at spatial infinity both these quantities can be identified with the on-shell Hamiltonian (with the lapse tending to unity).

The first term on the left-hand side (lhs) of Eq. (14) yields

$$\int_{\Sigma_{\mathcal{H}_\epsilon}} d^3x\sqrt{-g}\,\nabla_\mu(\Theta n^\mu + b^\mu) = \int_{\partial\Sigma_{\mathcal{H}_\epsilon}} d^2x\,\sqrt{\sigma}\,N\Theta$$
$$= \int_{\partial\Sigma_{\mathcal{H}_\epsilon}} d^2x\,\sqrt{\sigma}\,NK - \int_{\partial\Sigma_{\mathcal{H}_\epsilon}} d^2x\,\sqrt{\sigma}\,Nn_\mu a^\mu \quad , \tag{15}$$

where we used the identity:

$$\Theta = \Theta^t_t + (\Theta^\theta_\theta + \Theta^\varphi_\varphi) = -n_\mu a^\mu + K \quad , \tag{16}$$

which holds here since the spacelike normal n^μ is orthogonal to u^μ at the intersection of Σ with ${}^3\mathsf{B}$ [7]. Above, $a^\mu \equiv u^\nu\nabla_\nu u^\mu$ is the acceleration of the timelike hypersurface normal u^μ. The result (15) is a difference of two terms. The first term is 8π times the unreferenced quasilocal mass [1] [17,7] evaluated at infinity minus the unreferenced quasilocal mass evaluated near $\mathcal{H}$. The second term, when divided by 4π, is the difference between the Komar mass at infinity and the Komar mass at $\mathcal{H}_\epsilon$.

The second term on the lhs of (14) can be interpreted as follows. Since ${}^3\mathsf{B}$ is the time evolution of a two-sphere embedded in Σ, its line element is given by Eq. (12) with $r = r_0$, where r_0 is a constant. Thus, we have $R = 2/r_0^2$ on ${}^3\mathsf{B}$. Hence,

$$\int_{\Sigma_{\mathcal{H}_\epsilon}} d^3x\sqrt{-g}\,R = 8\pi\int_{r_{\mathcal{H}_\epsilon}}^\infty dr_0\,N(r_0)/\lambda(r_0)\,. \tag{17}$$

Assuming the integrability of the integrand in the rhs above ensures that $N(r)dr/\lambda(r)$ is an exact differential on Σ, say, equal to $df(r)$. Then the above result is equal to

$$8\pi\int_{r_{\mathcal{H}_\epsilon}}^\infty dr\,df(r)/dr = 8\pi f(r)\Big|_{r_{\mathcal{H}_\epsilon}}^\infty\,. \tag{18}$$

On Σ, suppose $N(r)/\lambda(r)$ approaches unity when $r \to \infty$ and when $r \to r_{\mathcal{H}}$. Then $f(r) \to r$ in these two neighborhoods. (In fact, for spherically symmetric electrovac spacetimes, the Einstein field equations ensure that on Σ, we have $N(r) = \lambda(r)$ on solutions, for all r. Hence, the above assumptions are met there.) In such a case, $R = dK_0/dr$, where $K_0 = -2/r$ is the trace of the extrinsic curvature of a two-sphere when embedded isometrically (with respect to B embedded in Σ in the black hole spacetime) in a *flat* spatial slice. Then, combining the above results, Eq. (17) can be reexpressed as

$$\int_{\Sigma_{\mathcal{H}_\epsilon}} d^3x\sqrt{-g}\,R = 8\pi r\Big|_{r_{\mathcal{H}_\epsilon}}^\infty = -\int_{\partial\Sigma_{\mathcal{H}_\epsilon}} d^2x\,\sqrt{\sigma}K_0\,. \tag{19}$$

If the spacetime is asymptotically flat, then $1/8\pi$ times the rhs above is the appropriate term to be added to the unreferenced Brown-York quasilocal energy to obtain the (normalized) physical QLE.

Using the above results in Eq. (14), we obtain

$$\frac{1}{8\pi}\int_{\partial\Sigma_{\mathcal{H}_\epsilon}} d^2x\,\sqrt{\sigma}\,(NK - K_0) = E_\infty - M_{\mathcal{H}_\epsilon}\,. \tag{20}$$

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[1]See Eq. (26) below for a discussion of this concept.

In the limit of the interior two-surface, $\mathcal{H}_\epsilon$, approaching arbitrarily close to $\mathcal{H}$, the above equation implies

$$\lim_{\epsilon\to 0} \frac{1}{8\pi} \int_{\partial\Sigma_{\mathcal{H}_\epsilon}} d^2x \, \sqrt{\sigma} \, (NK - K_0) = E_\infty - M_{\mathcal{H}} \quad , \tag{21}$$

where $M_{\mathcal{H}}$ is the Komar mass at $\mathcal{H}$. The lhs above is the difference of contributions from a two-sphere at $r \to \infty$ and from a two-sphere arbitrarily close to $\mathcal{H}$. As one approaches $\mathcal{H}$ along Σ, the lapse N vanishes. For SSS spacetimes (12), even $K = -2\lambda/r$ vanishes on such a surface. Thus, the contribution from this surface is purely due to the reference term K_0 and is defined by the following limiting procedure:

$$E_{\mathcal{H}} = \lim_{\epsilon\to 0} E_{\mathcal{H}_\epsilon} \equiv \lim_{\epsilon\to 0} \int_{\mathcal{H}_\epsilon} d^2x \sqrt{\sigma} \, \frac{(NK - K_0)}{8\pi} = - \lim_{\epsilon\to 0} \int_{\mathcal{H}_\epsilon} d^2x \sqrt{\sigma} \, \frac{K_0}{8\pi} \quad , \tag{22}$$

which is just the QLE at that surface. Let us analyze the contribution from the boundary term at spatial infinity. Note that by an earlier assumption we have $N/\lambda \to 1$ as $r \to \infty$. Thus, $NK = -2\lambda^2/r$ as $r \to \infty$. However, λ obeys the fall-off condition (13). Consequently, the contribution to the lhs of (21) from infinity is

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\mathsf{B}} d^2x \, \sqrt{\sigma} \, (NK - K_0) = 2\lambda_1 = 2E_\infty \, . \tag{23}$$

Hence, the lhs of Eq. (21) is equal to $2E_\infty - E_{\mathcal{H}}$. With these simplifications Eq. (21) itself reduces to the required identity, Eq. (4).

Note that we have nowhere assumed the spacetime to be a solution of general relativity (except in interpreting the condition on the Ricci tensor, Eq. (9), in terms of the constraints posed on the matter stress tensor). However, the association of K with the quasilocal energy has a nice justification [7,8] provided the quasilocal two-surface B is taken to be embedded in such a solution. Then, E is just the on-shell Hamiltonian with the lapse set equal to 1.

The spatial volume of integration in Eq. (14) could have been limited to a different or smaller region of Σ. In such cases, interpreting the surface terms on the lhs of Eq. (14) as QLE is not always possible. In that sense the horizon and the spatial infinity are very special locations for evaluating these terms.

In the above proof it is assumed that the quantities $E_{\mathcal{H}}$, $M_{\mathcal{H}}$, and E_∞, which appear in the identity (4), are evaluated on a single leaf, Σ_t, of the foliation. We now show that the proof can be strengthened such that the identity is applicable even in situations where contributions from the neighborhood of $\mathcal{H}$, namely, $E_{\mathcal{H}}$ and $M_{\mathcal{H}}$, are evaluated on a leaf different from the one on which E_∞ is computed. To see that this generalization holds, let us construct the boundary ${}^3\mathsf{B}$ such that t^a always lies on it. Define

$$Q_t \equiv \frac{1}{8\pi} \int_{\mathsf{B}} d^2x \, \sqrt{\sigma} K u^\mu t_\mu \, . \tag{24}$$

Then the law of conservation for the matter stress tensor implies [7]

$$Q_t(\Sigma_{t''} \cap {}^3\mathsf{B}) - Q_t(\Sigma_{t'} \cap {}^3\mathsf{B}) = - \int_{{}^3\mathsf{B}} d^3x \sqrt{-\gamma} \, T^{\mu\nu} n_\mu t_\nu \quad , \tag{25}$$

which is zero when the source term $T^{\mu\nu}n_\mu t_\nu$ vanishes. In such a case,

$$-Q_t = \frac{1}{8\pi}\int_{\mathsf{B}} d^2x\sqrt{\sigma}NK \quad , \tag{26}$$

which is conserved under diffeomorphisms along the orbits of t^a on ${}^3\mathsf{B}$: it is termed as the quasilocal mass of the system. Moreover, when B is a sphere of constant curvature-radius in SSS spacetimes, then the above equation implies

$$-Q_t = NE \quad , \tag{27}$$

for N is independent of the coordinates on B. Finally, since the norm of t^a is fixed along its integral curves, the quasilocal energy E is constant on ${}^3\mathsf{B}$. This completes the generalization of our proof. We can, therefore, state our result in the form of the following theorem:

For asymptotically flat, spherically symmetric static solutions of GR, if the matter stress tensor obeys Eq. (10) and if, on a constant Killing-time hypersurface, the ratio $N(r)/\lambda(r)$ is integrable and approaches unity at $\mathcal{H}$ and at i^0, then the identity, Eq. (4), is obeyed, and is implied by the Gauss-Codacci condition Eq. (11).

We now specifically compute the quasilocal quantities that appear in Eq. (4) for the Reissner-Nordstrom (RN) spacetimes and show that the identity is obeyed. The corresponding metric and the electromagnetic field can be given as (see, eg., [18]):

$$\begin{aligned} ds^2 = {} & [C_\theta Z_r - C_r Z_\theta]\left\{\frac{dr^2}{\Delta_r} + \frac{\sin^2\theta}{\Delta_\theta}\right\} \\ & + \frac{\Delta_\theta[C_r dt - Z_r d\varphi]^2 - \Delta_r[C_\theta dt - Z_\theta d\varphi]^2}{[C_\theta Z_r - C_r Z_\theta]} \quad , \end{aligned} \tag{28a}$$

$$F = \frac{2Q}{r^2}dr\wedge dt - 2P\sin\theta d\theta\wedge d\varphi \quad , \tag{28b}$$

where Q and P are the electric and magnetic monopole charges, respectively, of the hole. Also, t and r are the curvature coordinates.

The electromagnetic stress tensor is

$$8\pi T_{ab} = (E^2+B^2)\{\omega_a^{(0)}\omega_b^{(0)} + \omega_a^{(3)}\omega_b^{(3)} + \omega_a^{(2)}\omega_b^{(2)} - \omega_a^{(1)}\omega_b^{(1)}\} \quad , \tag{29}$$

where $E = Q/r^2$ and $B = P/r^2$ and the tetrad of forms are

$$\omega^{(0)} = \sqrt{\frac{\Delta_r}{C_\theta Z_r - C_r Z_\theta}}[C_\theta dt - Z_\theta d\varphi]\,, \qquad \omega^{(1)} = \sqrt{\frac{C_\theta Z_r - C_r Z_\theta}{\Delta_r}}dr \quad , \tag{30a}$$

$$\omega^{(2)} = \sqrt{\frac{C_\theta Z_r - C_r Z_\theta}{\Delta_\theta}}\sin\theta d\theta\,, \qquad \omega^{(3)} = \sqrt{\frac{\Delta_\theta}{C_\theta Z_r - C_r Z_\theta}}[C_r dt - Z_r d\varphi]\,. \tag{30b}$$

For RN spacetimes $C_\theta = 1$, $C_r = 0$, $Z_r = r^2$, $Z_\theta = 0$, $\Delta_r = r^2 - 2Mr + Q^2 + P^2$, and $\Delta_\theta = \sin^2\theta$, where M is the mass of the spacetime.

The Ricci tensor is given by

$$\mathcal{R}_{ab} = 8\pi T_{ab} \quad , \tag{31}$$

which is just Einstein's equation for $\mathcal{R} = 0$. It is clear from Eq. (29) that

$$\hat{t}^\mu \hat{t}^\nu \mathcal{R}_{\mu\nu} = -\hat{r}^\mu \hat{r}^\nu \mathcal{R}_{\mu\nu} \quad , \tag{32}$$

which is the condition (9). Moreover, the metric (28a), when applied to RN spacetimes, has the same form as Eq. (12). Hence the proof presented for SSS spacetimes remains valid in this case.

The characterizing relation (4) should hold good in all coordinate systems for which the spacelike hypersurfaces corresponding to constant coordinate-time foliate $\mathcal{H}$ and spatial infinity in a manner identical to the curvature coordinates. Let us in particular verify it for an electrically charged hole in isotropic coordinates. The metric in these coordinates is

$$ds^2 = -\left[\frac{1-\alpha^2/4r^2}{1+\frac{M}{r}+\frac{\alpha^2}{4r^2}}\right]^2 dt^2 + (1+M/r+\alpha^2/4r^2)^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] \quad , \tag{33}$$

where $\alpha^2 = M^2 - Q^2 - P^2$. For the metric (33), the energy and charge expressions are:

$$E = M + \alpha^2/2r \tag{34}$$

and

$$M_c = \frac{M + \alpha^2/r + M\alpha^2/4r^2}{1 + M/r + \alpha^2/4r^2}. \tag{35}$$

Hence, the relation (4) once again gives the (outer) horizon to be at $r = \alpha/2$, showing that it is valid for isotropic coordinates as well.

It is interesting to note that the two cases, $Q = 0 = P$ and $Q^2 + P^2 = M^2$, characterize conservation of charge and energy, respectively. The former is a special case of RN spacetimes; it corresponds to the Schwarzschild black hole ($Q = 0 = P$) spacetime. In this case, the gravitational charge obeys $M_c = M$. It is a conserved quantity: the value of M_c is independent of the leaf Σ, or even the location of a two-surface B (lying outside the horizon) on which it is evaluated. The latter case describes an extremal RN black hole spacetime. It follows from the above expressions for QLE and gravitational charge that the identity (4) holds even for such spacetimes. For an extremal hole it follows from Eq. (34) that $E = M$ everywhere, implying that it is a conserved quantity in the above sense. It also means that there is no "force" to drive the collapse and, hence, an extremal hole can never form from the collapse of dispersed matter distributions [19].

The difference between gravitational charge and QLE can be appreciated by considering the particular case of the RN spacetimes. Note that the gravitational charge is determined by the "gravitational charge density" defined as

$$\rho_c = T_0^0 - T_i^i = (1/4\pi)\mathcal{R}_{\mu\nu}u^\mu u^\nu \quad , \tag{36}$$

where $u_\mu u^\mu = -1$ and i is a space index. This should be distinguished from the matter-energy density, which is $T_0^0 = \rho$. In the case of the RN black hole $\rho = (Q^2 + P^2)/2r^4$, while $\rho_c = (Q^2 + P^2)/r^4$. In the Newtonian approximation, the QLE, $E(r)$, is the sum of matter energy density plus the (gravitational and electromagnetic) potential energy required

to build a ball of fluid by bringing its constituents together from a boundary of radius r. Furthermore, the contribution to $E(r_0)$ from the region $r > r_0$ is equal to $(Q^2 + P^2)/2r_0$, which is due to the electromagnetic field, plus $-M^2/2r_0$, which is due to gravity. Hence the energy enclosed by the region is $M - ((Q^2 + P^2)/2r_0 - M^2/2r_0)$. This is what E is, as given by (34) (with $r = r_0$ there), in the first approximation.

It would be interesting to explore if the identity (4) is valid for the Kerr-Newman spacetimes, atleast for certain choices of spacetime foliations and quasilocal two-boundaries. Unfortunately, in this case the exact expressions for QLE are not available, except when the two-boundary is taken to be at spatial infinity. However, Martinez [20] has evaluated QLE for constant stationary time Kerr slices bounded by different types of two-boundaries in the *slow rotation* approximation. The status of our identity in this approximate case is being studied [21].

In reality a black hole always sits in a cosmological background. It is therefore desirable to consider a black hole spacetime that is non-static (expanding) and asymptotically FRW. Just as we proved the identity (4) generically for asymptotically flat SSS spacetimes earlier, one can similarly prove it to hold for certain asymptotically non-flat spacetimes as well [21]. Here, we will briefly demonstrate its validity for some of these spacetimes. Consider the solution describing an asymptotically FRW, electrically and magnetically charged black hole [22,23]:

$$ds^2 = -\frac{F^2}{G^2}dt^2 + S(t)^2 G^2 H^{-2}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] \tag{37}$$

where

$$F = 1 - H\alpha^2/(4r^2S^2) \tag{38a}$$
$$H = 1 + kr^2/4,\ \alpha^2 = M^2 - Q^2 - P^2 \tag{38b}$$
$$G = 1 + H^{1/2}M/(rS) + H\alpha^2/(4r^2S^2) \quad , \tag{38c}$$

where $k = \pm 1, 0$, is the space curvature parameter and $S(t)$ is the scale factor. Above, we have chosen to write the metric in isotropic coordinates, which in this case also happens to be comoving.

The QLE in these asymptotically FRW spacetimes is obtained by using Eq. (1), with the corresponding FRW spacetime (where $M = 0 = Q = P$) chosen as the reference. It is evaluated for the foliation comprising of t =constant hypersurfaces. This gives for the QLE:

$$E(r) = -Sr^2G'/H \quad , \tag{39}$$

in the isotropic gauge. Using the expression for G given in Eq. (38a), we get

$$E = MH^{-3/2} + \alpha^2/(2rSH) \quad , \tag{40}$$

where H and α are defined above. Here $E(\infty) = MH^{-3/2}$ as $rS \to \infty$. When we switch off the expansion, i.e., set $S = \mathit{const.}$, we get the energy for an electrically and magnetically charged black hole in the Einstein universe. For $S = 1, k = 0$, we recover the energy (34) in the isotropic coordinates. Thus, the above expression has the expected static limit.

Obtaining the gravitational charge for such non-static spacetimes is more subtle. Here we adopt a suitable generalization of Eq. (2). In that equation, we identify the lapse function as $N = \sqrt{-g_{tt}}$, where t is now the comoving time coordinate in the metric (37). Such a choice is motivated by the fact that as $k \to 0$ and $S \to 1$, the above cosmological metric (37) approaches the SSS metric (33). Consequently, the comoving time gets identified with the Killing time of the resulting static solution. Then, the gravitational charge in such a spacetime is given by

$$M_c = \alpha^2/2rSH + (MH^{-3/2} + \alpha^2/2rSH)F/G. \tag{41}$$

Note that $M_c(\infty) = E_\infty = MH^{-3/2}$ and Eq. (4) again defines the horizon at $rS = \alpha H^{1/2}/2$. Thus the black hole characterization (4) holds good for an electrically charged black hole sitting in an FRW expanding universe. Note that for $S = 1, k = 0$, we recover the gravitational charge (34) (with $P = 0$) in the isotropic coordinates.

Yet another case of applicability of the identity (4), is an SSS black hole spacetime with a global monopole charge in it [24,25,27]. The metric for the spacetime dual to the Schwarzschild black hole incorporating global monopole charge is given by

$$ds^2 = -\left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)dt^2 + \left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad , \tag{42}$$

where η represents the global monopole charge. Setting $\eta = 0$ yields the Schwarzschild solution. The event horizon is at $r = 2M/(1-8\pi\eta^2)$. The physical effects of global monopole charge have been studied in Refs. [25,26]. The spacetime is not vacuum and is asymptotically non-flat [15]. But it is a solution of the electrogravity-dual vacuum equation [27]. The QLE, on a foliation comprising of t =constant hypersurfaces, in the above spacetime is

$$E = r\left[(1 - 8\pi\eta^2)^{1/2} - \left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)^{1/2}\right] \quad , \tag{43}$$

where the Hawking-Horowitz prescription [8] mentioned before was used to compute the reference contribution of $r(1 - 8\pi\eta^2)^{1/2}$ above. The reference spacetime in this case turns out to be non-flat; rather it is the dual-flat solution in the same sense as the metric (42) is dual-vacuum. Here, $E_\infty = M(1 - 8\pi\eta^2)^{-1/2}$, $E_{\mathcal{H}} = 2E_\infty$. The gravitational charge is $M_c(r \to \infty) = E_\infty$. Again it is straightforward to show that Eq. (4) will define the location of the horizon.

In our discussions above of black hole solutions that are asymptotically FRW or that have a global monopole charge, we have demonstrated that our identity (4) is applicable to non-static as well as asymptotically non-flat cases. The details of the derivation of this identity for such spacetimes from a requirement of the type (8), will be given elsewhere [21].

Note that the identity (4) has the following implication on the non-attainment of extremality. The particular expression for the gravitational charge of a black hole, $M_{\mathcal{H}} = (\kappa/4\pi)A$, relates it to the surface gravity κ (and, therefore, the temperature) of the hole. Here A is area of the horizon. The third law of black hole dynamics states that it is impossible to reduce gravitational charge of a hole to zero by a finite sequence of physical processes [19]. In view of the relation (4), we could as well say that the magnitude of the field energy,

$|E_\infty - E_\mathcal{H}|$ cannot be reduced to zero in a finite sequence of physical interactions. Since the surface gravity of the RN hole is zero in the extremal limit, i.e., $M^2 = Q^2 + P^2$, the field energy is also zero in this case, which implies that an extremal hole can never be formed from the collapse of dispersed matter distributions. Similarly, a non-extremal hole can never turn extremal, say, due to infalling charged matter. Recent quantum field theoretic and topological considerations seem to suggest that the converse may also be true, i.e., extremal RN holes may also be prevented from turning into non-extremal ones (see Refs. [8,28]). In that case the extremal and non-extremal holes would be analogous to particles of zero and non-zero mass, respectively, where the gravitational charge acts as a nice analogue of mass [13].

We observe that the *horizon*-based quantity $(E_\mathcal{H} - M_\mathcal{H})$ is the analogue of the internal energy of the whole spacetime (up to the addition of an exact form) in the thermodynamical laws of static black holes. This is because E_∞ itself has the interpretation of being the thermodynamical internal energy of the whole spacetime [7].

It was known from the formulation of the laws of black hole mechanics that *variations* of certain quantities at the horizon and at infinity are related. Here, we have shown that a non-variational identity relating some of these quantities also exists in general relativity. In the past, Iyer and Wald have also explored similar possibilities [29]. These authors generalized the definition of the BY quasilocal mass to a more general class of diffeomorphism invariant Lagrangian theories of gravity. One of the results of their work is that as one approaches close to $\mathcal{H}$ along a smooth hypersurface Σ, which is transverse to the Killing time t^a, the Noether charge associated with t^a approaches twice the boundary terms in the gravitational action (which in turn depend on the choice of boundary conditions imposed on the dynamical fields). Our identity is similar in spirit to (but different in content from) this relation. We are currently studying the possibility of an identity similar to Eq. (4) existing in similar Lagrangian theories of gravity.

Finally, it is always welcome to gain some insight into the difficult and ambiguous concept of energy in GR. Most of the definitions refer to a quasilocal energy, which generally includes contribution of the matter energy density, while gravitational charge is essentially defined through the Komar integral or its generalizations. The latter is related to the formulation of the Gauss law for stationary spacetimes [10]. Here we have extended the application of these constructs to non-static and asymptotically non-flat spacetimes as well. Although we demonstrated that our identity (4) is applicable to these cases, after suitably adapting the Komar integral (3) for such examples, we did not derive it from an "embeddability" condition of the type (8). Moreover, even where we proved the identity, our consideration was limited essentially to eternal black hole spacetimes, where the apparent and event horizons overlapped. To be astrophysically relevant, however, one must deal with the case of isolated horizons [30]. Generalization of the proof for the applicability of an identity of the type (4), to other black hole spacetimes, and to the case of isolated horizons, is presently under consideration [21].

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